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ON A GENERALIZATION OF THE NAGHDI-HSU TRANSFORMATION AND ITS APPLICATION TO PROBLEMS OF ELASTICITY THEORY*

A.N. BORODACHEV

The following model is considered for the inhomogeneity of an elastic material: the shear modulus of the material is constant, but Poisson's-ratio (or the modulus of elasticity) depends in an arbitrary manner on three Cartesian coordinates. A linear transformation of vector fields is introduced, which is a generalization of the well-known Naghdi-Hsu transformation /1, 2/ (NHT) that enables the solution of the equilibrium equation of elasticity theory for bodies with variable Poisson's ratio to be represented in terms of a vector function (harmonic when there are no body forces), and proof of its completeness. In passing, a new variation of the NHT is formulated in which the integrals over the body volume are replaced by integrals over its surface. The fundamental solution of the equilibrium equation in displacements is presented in explicit form for an unbounded body with inhomogeneity of the type under consideration.

Some problems in this area were investigated in /3-5/.

1. Let an isotropic elastic material occupy a bounded regular domain V with boundary S of a three-dimensional real Euclidean space R^3 whose points will be denoted by $\mathbf{x} = (x_1, x_2, x_3)$. The shear modulus $\mu > 0$ of the elastic material is constant while Poisson's ratio $\nu(\mathbf{x})$ is an arbitrary function from the class C^1 that satisfies the standard conditions $-1 < \nu(\mathbf{x}) < {}^1/_2$ /6/. In this case the modulus of elasticity of the material $E(\mathbf{x}) = 2\mu[1 + \nu(\mathbf{x})]$ is a positive-definite function of the coordinates.

We will use the following notation: $\mathbf{u}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$ are the displacement and body force vectors, and Δ and ∇ are the Laplace and gradient opertors in R^3 and $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

The equilibrium equation in displacements for this model of elastic material inhomogeneity has the form

$$L\mathbf{u}(\mathbf{x}) = -\mu^{-1}\mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in V \tag{1.1}$$
$$L\mathbf{u} \equiv \Delta\mathbf{u} + \nabla(\eta\nabla\cdot\mathbf{u}), \quad \eta(\mathbf{x}) = [1 - 2\nu(\mathbf{x})]^{-1}$$

Applying the divergence and curl operations to (1.1), we conclude that if $\nabla\cdot\mathbf{F} = 0$ and $\nabla\times\mathbf{F} = 0$, the functions $(1 + \eta)\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$ are harmonic. In the case of a homogeneous material the function $\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$ are harmonic, as is well-known /6/.

Eq.(1.1) is satisfied identically if we set

$$\mathbf{u}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) + \frac{1}{8\pi}\nabla\int_V \frac{\gamma(\mathbf{y})\nabla\cdot\mathbf{B}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{1.2}$$
$$\gamma(\mathbf{x}) = [1 - \nu(\mathbf{x})]^{-1}$$
$$\Delta\mathbf{B}(\mathbf{x}) = -\mu^{-1}\mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in V$$

where the second component in the first relationship of (1.2) is the gradient of the particular solution of Poisson's equation $\Delta b = -{}^1/_2\gamma\nabla\cdot\mathbf{B}$, $\mathbf{x} \in V$.

Another approach to the construction of the solution of (1.1) in terms of four scalar functions is described in /7/.

In the case of a homogeneous material, relationships (1.2) reduce to the well-known Naghdi-Hsu solution /1/ whose relation to the Papkovich-Neuber and Galerkin representations is established in /2/.

Furthermore, we introduce a linear transformation generalizing the NHT /2/ and closely associated with (1.1). In particular, its utilization enables us to prove the completeness of representation (1.2).

Lemma. An arbitrary vector field $\mathbf{N}$ in V allows a unique representation of the type

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$$\mathbf{N}(\mathbf{x})=\mathbf{H}(\mathbf{x})+\frac{1}{8\pi}\nabla\int_V\frac{\gamma(\mathbf{y})\nabla\cdot\mathbf{H}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{1.3}$$

where

$$\mathbf{H}(\mathbf{x})=\mathbf{N}(\mathbf{x})-\frac{1}{4\pi}\nabla\int_V\frac{\eta(\mathbf{y})\nabla\cdot\mathbf{N}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{1.4}$$

Conversely, if $\mathbf{H}$ is an arbitrary vector field in V, then it allows of a unique representation of the type (1.4), where $\mathbf{N}$ is given by (1.3).

Proof. Applying the divergence operation to (1.4) by using the relationship /8/

$$\Delta\int_V\varphi(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{-1}dV(\mathbf{y})=-4\pi\varphi(\mathbf{x}) \tag{1.5}$$

where φ is an arbitrary function, we find $\nabla\cdot\mathbf{N}=(2\eta)^{-1}\gamma\nabla\cdot\mathbf{H}$. Substituting this expression into (1.4), we obtain the representation (1.3).

To prove uniqueness, we assume that there are two representations of the type (1.3): $\mathbf{H}_1$ and $\mathbf{H}_2$. Then by forming the difference of the right-hand sides of (1.3) and evaluating its divergence by using (1.5), we find $\nabla\cdot(\mathbf{H}_1-\mathbf{H}_2)=0$, after which we conclude that $\mathbf{H}_1=\mathbf{H}_2$.

The proof of the converse assertion of the lemma is performed analogously.

Relationships (1.3) and (1.4), connecting the vector fields $\mathbf{N}$ and $\mathbf{H}$, will be called the generalized Naghdi-Hsu transformation (GNHT).

Let us define the spaces $N=\{\mathbf{N}(\mathbf{x}):\ L\mathbf{N}(\mathbf{x})=-\mu^{-1}\mathbf{F}(\mathbf{x}),\ \mathbf{x}\in V\}$ and $H=\{\mathbf{H}(\mathbf{x}):\ \Delta\mathbf{H}(\mathbf{x})=-\mu^{-1}\mathbf{F}(\mathbf{x}),\ \mathbf{x}\in V\}$. The relation of the GNHT to (1.1) is established by the following.

Theorem. The GNHT (1.3) and (1.4) establish a one-to-one correspondence between the spaces N and H.

Proof. Acting on (1.3) with the operator L or on (1.4) with Δ we obtain the identity $L\mathbf{N}(\mathbf{x})\equiv\Delta\mathbf{H}(\mathbf{x}),\ \mathbf{x}\in V$. The proof of the theorem is completed by using the lemma and this identity.

As a consequence, we conclude from the theorem that the representation of the solution (1.1) in the form (1.2) is complete. Moreover, the vector function $\mathbf{B}$, that is harmonic when there are no bulk forces, is defined in a unique manner by the displacement field

$$\mathbf{B}(\mathbf{x})=\mathbf{u}(\mathbf{x})-\frac{1}{4\pi}\nabla\int_V\frac{\eta(\mathbf{y})\,\nabla\cdot\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{1.6}$$

Evaluating the divergence of curl of (1.2) or (1.6) we find

$$(1+\eta)\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{B},\quad \nabla\times\mathbf{u}=\nabla\times\mathbf{B}$$

By using the formulas connecting the stress and displacement /7/ and (1.2), we obtain a representation of the stress tensor components in terms of the vector function $\mathbf{B}$ (δ_{ij} is the Kronecker delta)

$$\mu^{-1}\sigma_{ij}(\mathbf{x})=\delta_{ij}\nu(\mathbf{x})\gamma(\mathbf{x})\nabla\cdot\mathbf{B}(\mathbf{x})+\frac{\partial B_i(\mathbf{x})}{\partial x_j}+ \frac{\partial B_j(\mathbf{x})}{\partial x_i}+\frac{1}{4\pi}\frac{\partial^2}{\partial x_i\partial x_j}\int_V\frac{\gamma(\mathbf{y})\,\nabla\cdot\mathbf{B}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{1.7}$$

2. In the case of a homogeneous material when $\nu(\mathbf{x})=\nu\equiv\text{const}$, Eqs. (1.2) and (1.6) reduce to the well-known NHT /2/

$$\mathbf{u}(\mathbf{x})=\mathbf{B}(\mathbf{x})+\frac{1}{8\pi(1-\nu)}\nabla\int_V\frac{\nabla\cdot\mathbf{B}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}) \tag{2.1}$$

$$\mathbf{B}(\mathbf{x})=\mathbf{u}(\mathbf{x})-\frac{1}{4\pi(1-2\nu)}\nabla\int_V\frac{\nabla\cdot\mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y})$$

We will next show that the volume integrals in relationships (2.1) can be replaced by integrals over the surface S when there are no bulk forces. Indeed, we have for an arbitrary harmonic function $\varphi(\mathbf{x})$ /9/ ($\mathbf{n}$ is the external normal to the surface S)

$$\int_V\frac{\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y})=\frac{1}{2}\int_S\left[\varphi(\mathbf{y})\frac{\partial|\mathbf{x}-\mathbf{y}|}{\partial\mathbf{n}}-\frac{\partial\varphi(\mathbf{y})}{\partial\mathbf{n}}|\mathbf{x}-\mathbf{y}|\right]dS(\mathbf{y}) \tag{2.2}$$

Taking into account that the functions $\nabla\cdot\mathbf{u}$ and $\nabla\cdot\mathbf{B}$ are harmonic V for $F=0$, and using (2.2), we reduce (2.1) to the form

$$\mathbf{u}(\mathbf{x})=\mathbf{B}(\mathbf{x})+\frac{1}{16\pi(1-\nu)}\nabla\int_S\left\{[\nabla\cdot\mathbf{B}(\mathbf{y})]\frac{\partial|\mathbf{x}-\mathbf{y}|}{\partial\mathbf{n}}-\frac{\partial[\nabla\cdot\mathbf{B}(\mathbf{y})]}{\partial\mathbf{n}}|\mathbf{x}-\mathbf{y}|\right\}dS(\mathbf{y}) \tag{2.3}$$

$$\mathbf{B}(\mathbf{x})=\mathbf{u}(\mathbf{x})-\frac{1}{8\pi(1-2\nu)}\nabla\int_S\left\{[\nabla\cdot\mathbf{u}(\mathbf{y})]\frac{\partial|\mathbf{x}-\mathbf{y}|}{\partial\mathbf{n}}-\frac{\partial[\nabla\cdot\mathbf{u}(\mathbf{y})]}{\partial\mathbf{n}}|\mathbf{x}-\mathbf{y}|\right\}dS(\mathbf{y})$$

Relationships (2.3) are a new variation of the NHT in which the integrals over the body volume V are replaced by integrals over its surface S.

As already noted, the vector function $\mathbf{B}$ is defined in a unique manner by the displacement field. As is shown in /2/, this enables us to use the NHT (2.1) in the solution of specific problems by the finite element method. Application of the Papkovich-Neuber functions for these purposes is not possible /10/ since they are not defined in a unique manner by the displacement field. In this connection we note that the NHT modification (2.3) can apparently turn out to be useful in solving boundary value problems by the method of boundary elements /11/ whose basis is the reduction of the problem to integral equations over the body surface.

3. Up to now it was assumed that the domain V is bounded. For unbounded domains the representation of the solution (1.2) remains valid if a gradient operator is introduced under the integral sign. We consequently obtain

$$\mathbf{u}(\mathbf{x})=\mathbf{B}(\mathbf{x})+\frac{1}{8\pi}\int_V\gamma(\mathbf{y})\nabla\cdot\mathbf{B}(\mathbf{y})\nabla_x(|\mathbf{x}-\mathbf{y}|^{-1})\,dV(\mathbf{y}) \tag{3.1}$$

or taking account of the relationship $\nabla_x(|\mathbf{x}-\mathbf{y}|^{-1})=-\nabla_y(|\mathbf{x}-\mathbf{y}|^{-1})$

$$\mathbf{u}(\mathbf{x})=\mathbf{B}(\mathbf{x})-\frac{1}{8\pi}\int_V\gamma(\mathbf{y})\nabla\cdot\mathbf{B}(\mathbf{y})\nabla_y(|\mathbf{x}-\mathbf{y}|^{-1})\,dV(\mathbf{y}) \tag{3.2}$$

Indeed $\gamma(\mathbf{y})=O(1)$, $\nabla\cdot\mathbf{B}(\mathbf{y})=O(|\mathbf{y}|^{-2})$, $|\mathbf{x}-\mathbf{y}|^{-1}=O(|\mathbf{y}|^{-1})$, $\nabla(|\mathbf{x}-\mathbf{y}|^{-1})=O(|\mathbf{y}|^{-2})$ for $|\mathbf{y}|\to\infty$ and the integrands in (3.1) and (3.2) are of the order of $|\mathbf{y}|^{-4}$ as $|\mathbf{y}|\to\infty$ (unlike the integrand in (1.2) which is of the order of $|\mathbf{y}|^{-3}$ as $|\mathbf{y}|\to\infty$) so that the integrals in (3.1) and (3.2) converge.

The representation (3.2) (or (3.1)) enables the fundamental solution of the equilibrium equation for an unbounded elastic body whose shear modulus is constant while Poisson's ratio (or elastic modulus) depends in an arbitrary manner on the three Cartesian coordinates, to be constructed explicitly.

Let $V=R^3$ and a let a unit concentrated force direction parallel to the Ox_k axis act at a certain point $\xi\in R^3$. In this case, the components of the bulk force vector have the form

$$F_i=\delta(\mathbf{x}-\xi)\,\delta_{ik}\quad(i,k=1,2,3)$$

where $\delta(\mathbf{x})$ is the three-dimensional Dirac function and the vector function $\mathbf{B}^{(k)}$ is determined from the equations

$$\Delta_x B_i^{(k)}(\mathbf{x},\xi)=-\mu^{-1}\delta(\mathbf{x}-\xi)\,\delta_{ik}$$

whose particular solutions that decay at infinity have the form

$$B_i^{(k)}(\mathbf{x},\xi)=\frac{\delta_{ik}}{4\pi\mu|\mathbf{x}-\xi|} \tag{3.3}$$

By using (3.3) we calculate

$$\nabla_x\cdot\mathbf{B}^{(k)}(\mathbf{x},\xi)=\frac{1}{4\pi\mu}\frac{\partial}{\partial x_k}(|\mathbf{x}-\xi|^{-1}) \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) we obtain a representation for the displacement field caused by a unit concentrated force applied at a point ξ and directed parallel to the Ox_k axis

$$u_i^{(k)}(\mathbf{x},\xi)=\frac{\delta_{ik}}{4\pi\mu|\mathbf{x}-\xi|}-\frac{v_{ik}(\mathbf{x},\xi)}{32\pi^2\mu} \tag{3.5}$$
$$v_{ik}(\mathbf{x},\xi)=\int_{R^3}\gamma(\mathbf{y})\frac{\partial}{\partial y_k}(|\mathbf{y}-\xi|^{-1})\frac{\partial}{\partial y_i}(|\mathbf{x}-\mathbf{y}|^{-1})\,dV(\mathbf{y})$$

The improper integrals in (3.5) converge for $\mathbf{x}\neq\xi$ and have a singularity of the type $|\mathbf{x}-\xi|^{-1}$ at the point $\mathbf{x}=\xi$. Therefore, as in the case of a homogeneous material, the fundamental solution (3.5) has a singularity of order $|\mathbf{x}-\xi|^{-1}$ at the point of application of the force.

For $\nu(\mathbf{x}) = \nu \equiv \text{const}$ the functions v_{ik} are evaluated explicitly by using the three-dimensional Fourier integral transform

$$v_{ik}(\mathbf{x}, \boldsymbol{\xi}) = \frac{2\pi}{1-\nu} \frac{\partial^2 |\mathbf{x}-\boldsymbol{\xi}|}{\partial x_i \partial x_k}$$

We hence obtain the known representation of the fundamental solution of the equilibrium equation of homogeneous elasticity theory, the Kelvin matrix /6/.

We note that by using relationships (1.7), (3.3) and (3.4), the stress field caused by the action of a unit concentrated force on an unbounded elastic body with inhomogeneity of the type consideration can be evaluated.

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ON CERTAIN FORMULATIONS OF THE BOUNDARY-ELEMENTS METHOD*

V.YA. TERESHCHENKO

Variational formulations are proposed for the boundary-element method (BEM) to solve linear problems of elasticity theory with a known Green's tensor. Unlike existing BEM formulations utilizing the method of weighted residuals /1/ or boundary integral equations /2/, the formulations to be considered below use a variational formulation of the problems for boundary functionals /3/ in a set of allowable functions in the form of double-layer potentials whose density is given in the form of BEM basis functions. Also examined is a BEM variational formulation on the basis of minimization problems for Trefftz generalized functionals of the fundamental boundary value problems of linear elasticity theory /4/. A basis for the formulations is presented. Utilization of the proposed BEM formulations is effective in solving boundary-contact problems; consequently, a numerical realization is examined with an example of a unilateral variational problem (of the generalized Signorini problem type /5/) corresponding to the classical contact problem of inserting a stamp

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